

Graphs on the Torus and Geometry of Numbers

ALEXANDER SCHRIJVER

*CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands and
Department of Mathematics, University of Amsterdam,
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands*

Received October 10, 1991

We show that if G is a graph embedded on the torus S and each non-nullhomotopic closed curve on S intersects G at least r times, then G contains at least $\lfloor \frac{3}{4}r \rfloor$ pairwise disjoint nonnullhomotopic circuits. The factor $\frac{3}{4}$ is best possible. We prove this by showing the equivalence of this bound to a bound in the two-dimensional geometry of numbers. To show the equivalence, we study *integer norms*, i.e., norms $\|\cdot\|$ such that $\|x\|$ is an integer for each integer vector x . In particular, we show that each integer norm in two dimensions has associated with it a graph embedded on the torus, and conversely. © 1993 Academic Press, Inc.

1. DESCRIPTION OF RESULTS

Call a closed curve on a surface S *nontrivial* if it is not nullhomotopic. For any graph G embedded on S , the *representativity* (or *face width*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all nontrivial closed curves on S . We will show the following theorem.

THEOREM 1. (i) *Any graph G embedded on the torus contains at least $\lfloor \frac{3}{4}r(G) \rfloor$ pairwise disjoint nontrivial circuits.*

(ii) *The factor $\frac{3}{4}$ is best possible.*

Here $\lfloor x \rfloor$ denotes the lower integer part of x . A *circuit* is a simple closed curve contained in G . This is related to a result of Brunet, Mohar, and Richter [3] showing that any graph G embedded on any compact orientable surface S contains $\lfloor \frac{1}{2}(r(G) - 1) \rfloor$ pairwise disjoint nontrivial circuits.

Remark 1. The representativity of a graph embedded on a surface is recently a focus of attention in the study of minimal genus embeddings of graphs and of graph minors and disjoint paths (see [1, 9, 14–16, 24]). In particular, Robertson and Seymour [14] showed:

for any compact surface S and any graph H embedded on S there exists a number r so that any graph G embedded on S with representativity at least r contains H as a minor. (1)

In fact, Robertson and Seymour showed that for any graph H embedded on a compact surface S such that each vertex of H has degree at most three, there exists a number r with the property: for each graph G embedded on S with representativity at least r there exists a homeomorphism $\phi: S \rightarrow S$ such that $\phi[H] \subseteq G$. (This implies (1).)

One of the simplest special cases is that for each natural number k there exists a number $r(k)$ such that any graph G embedded on the torus S with representativity at least $r(k)$ contains k pairwise disjoint nontrivial circuits. Theorem 1 asserts that we can take $r(k) = \lceil \frac{4}{3}k \rceil$, where $\frac{4}{3}$ is the best possible factor.

We will show that Theorem 1 is equivalent to the following result in the geometry of numbers: For any symmetric convex body K (i.e., full dimensional compact convex set K with $K = -K$) in \mathbb{R}^n , let

$$K^* := \{y \in \mathbb{R}^n \mid y^T x \leq 1 \text{ for each } x \in K\}. \quad (2)$$

As is well known, K^* is again a symmetric convex body, and $(K^*)^* = K$. Now Theorem 1 is equivalent to:

THEOREM 2. (i) *For any symmetric convex body K in \mathbb{R}^2 , there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$*
 (ii) *The factor $\frac{4}{3}$ is best possible.*

Although we assume that this result belongs to the folklore of the geometry of numbers, we were not able to locate a proof in the literature (The best result in this direction we found in the literature was by Mahle [12] who proved a factor $\sqrt{2}$ replacing $\frac{4}{3}$ in Theorem 2(i).) Therefore, for completeness we describe a proof of Theorem 2 in Section 3 of this paper.

Remark 2. As is well known (cf. Cassels [4], Lekkerkerker [11]), there are several equivalent forms for Theorem 2(i). First, for any symmetric convex body K in \mathbb{R}^2 not containing a nonzero integer vector, there exists a nonzero integer vector c such that $c^T x \leq \frac{4}{3}$ for each vector x in K .

Second, define for each symmetric convex body K , $\lambda(K)$ to be the smallest value of λ for which $\lambda \cdot K$ contains a nonzero integer vector. Then for any symmetric convex body K in \mathbb{R}^2 , $\lambda(K) \cdot \lambda(K^*) \leq \frac{4}{3}$.

Third, for any norm $\|\cdot\|$ in \mathbb{R}^n , the dual norm $\|\cdot\|_*$ is defined by

$$\|y\|_* := \sup_x \frac{y^T x}{\|x\|}, \quad (3)$$

where the supremum ranges over all nonzero vectors x in \mathbb{R}^n . Then, for any norm $\|\cdot\|$ in \mathbb{R}^2 , there exist nonzero integer vectors x and y such that $\|x\| \cdot \|y\|_* \leq \frac{4}{3}$.

The equivalence of Theorems 1 and 2 is proved with the help of the following theorem given in [19] ([21, 6] gave more direct proofs and extensions to the directed case). Call two closed curves on the torus *freely homotopic* if one can be shifted continuously to the other over the torus (so there is no fixed “base point”).

For any two closed curves C and D , let $\text{mincr}(C, D)$ denote the minimum number of crossings of C' and D' (counting multiplicities), where C' and D' range over all closed curves freely homotopic to C and D , respectively.

THEOREM 3. *Let G be a graph embedded on the torus S , and let C be a simple closed curve on S . Then G contains k pairwise disjoint circuits each freely homotopic to C , if and only if each closed curve D on S has at least $k \cdot \text{mincr}(C, D)$ intersections with G (counting multiplicities).*

To describe the equivalence of Theorems 1 and 2, represent the torus as the product $S^1 \times S^1$, where S^1 is the unit circle in the complex plane. For each $(m, n) \in \mathbb{Z}^2$, let $C_{m,n}: S^1 \rightarrow S^1 \times S^1$ be the closed curve on the torus defined by

$$C_{m,n}(z) := (z^m, z^n), \tag{4}$$

for $z \in S^1$.

Now, as is well known (cf. [23, Section 6.2.2]), the $C_{m,n}$ form a system of representatives for the free homotopy classes of closed curves on the torus. Moreover,

$$\text{mincr}(C_{m,n}, C_{m',n'}) = |mn' - m'n| \tag{5}$$

for all $m, n, m', n' \in \mathbb{Z}$.

Let G be a graph on the torus, such that each face of G is an open disk, i.e., such that $r(G) > 0$. (This clearly will be no restriction in Theorem 1.) Define for each $(m, n) \in \mathbb{Z}^2$, $f_G(m, n)$ as the minimum number of intersections of C' and G (counting multiplicities), where C' ranges over all closed curves homotopic to $C_{m,n}$.

It is not difficult to see that

$$\begin{aligned} \text{(i)} \quad & f_G(m + m', n + n') \leq f_G(m, n) + f_G(m', n'), \\ \text{(ii)} \quad & f_G(km, kn) = |k| \cdot f_G(m, n) \end{aligned} \tag{6}$$

hold for all $(m, n), (m', n') \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. (The inequality in (i) follows from the fact that if C is freely homotopic to $C_{m,n}$ and C' is freely homotopic to $C_{m',n'}$ and (m, n) and (m', n') are linearly independent, then C has a crossing with C' . We can concatenate C and C' at this crossing so

as to obtain a closed curve C'' freely homotopic to $C_{m+m',n+n'}$ with $\text{cr}(G, C'') = \text{cr}(G, C) + \text{cr}(G, C')$, where cr denotes the number of crossings. The equality in (ii) is easy.) Hence there exists a unique norm $\|\cdot\|$ in \mathbb{R}^2 with the property that $\|(m, n)\| = f_G(m, n)$ for each $(m, n) \in \mathbb{Z}^2$.

Having this, we give one half of the proof of the equivalence of Theorems 1 and 2:

Proof of the implications Theorem 2(i) \Rightarrow Theorem 1(i) and Theorem 1(ii) \Rightarrow Theorem 2(ii). The representativity $r(G)$ of G is equal to the minimum of $\|(m, n)\|$ over all nonzero integer vectors (m, n) . Hence, by Theorem 2(i) (third variant in Remark 2), there exists a nonzero integer vector (m', n') such that $\|(m', n')\|_* \leq \frac{4}{3}r(G)^{-1}$.

By definition (3) of $\|\cdot\|_*$,

$$\frac{(m', n')^T (m, n)}{\|(m, n)\|} \leq \frac{4}{3r(G)} \quad (7)$$

for each nonzero vector (m, n) in \mathbb{R}^2 . This implies

$$\frac{3}{4}r(G)|m'm + n'n| \leq f_G(m, n) \quad (8)$$

for each integer vector (m, n) . Therefore, as $|m'm + n'n| = \text{mincr}(C_{m,n}, C_{n',-m'})$, by Theorem 3, G contains $\lfloor \frac{3}{4}r(G) \rfloor$ pairwise disjoint circuits, each freely homotopic to $C_{n',-m'}$. This shows Theorem 1(i).

This construction also shows that Theorem 1(ii) implies Theorem 2(ii), since any better factor in 2(i) would imply a better factor in 1(i). ■

To see the other implications, we consider integer norms. We call a norm $\|\cdot\|$ in \mathbb{R}^n an *integer norm* if $\|x\|$ is an integer for each x in \mathbb{Z}^n .

Above we saw that each graph G embedded on the torus gives an integer norm $\|\cdot\|$ in \mathbb{R}^2 such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n) . In fact each integer norm in \mathbb{R}^2 can be constructed in this way:

THEOREM 4. *For each integer norm $\|\cdot\|$ in \mathbb{R}^2 there exists a graph G embedded on the torus such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n) .*

We will give a proof of this theorem in Section 2 below.

Proof of the implications Theorem 1(i) \Rightarrow Theorem 2(i) and Theorem 2(ii) \Rightarrow Theorem 1(ii). We first show the first implication. Let K be a symmetric convex body in \mathbb{R}^2 not containing any nonzero integer vector. We show that $\frac{4}{3} \cdot K^*$ contains a nonzero integer vector.

We may assume that K is a polygon in \mathbb{R}^2 with rational vertices (since we can make K slightly larger). Then also K^* is a polygon with rational vertices.

Define the norm $\|\cdot\|$ in \mathbb{R}^2 by

$$\|x\| := \min\{\lambda \mid x \in \lambda \cdot K\} = \max\{x^T y \mid y \in K^*\} \tag{9}$$

for $x \in \mathbb{R}^2$. Let t be a common multiple of the denominators of the components of the vertices of K^* , with the further property that t is also a multiple of four.

Then $t \cdot \|\cdot\|$ is an integer norm, as the maximum in (9) is attained at a vertex of K^* . Hence, by Theorem 4, there exists a graph G embedded on the torus such that $f_G(m, n) = t \cdot \|(m, n)\|$ for each integer vector (m, n) .

As K contains no nonzero integer vector, we know that $\|(m, n)\| > 1$ for each nonzero integer vector (m, n) , and hence $f_G(m, n) > t$ for each nonzero integer vector (m, n) . So the representativity $r(G)$ of G is larger than t .

By Theorem 1(i), G contains $\frac{3}{4}t$ pairwise disjoint nontrivial circuits. They all are mutually freely homotopic; say, they are all freely homotopic to $C_{m,n}$. So, by the necessity of the condition in Theorem 3 and by (5), for each integer vector (m', n') ,

$$\frac{3}{4}t \cdot |mn' - m'n| = \frac{3}{4}t \cdot \text{mincr}(C_{m,n}, C_{m',n'}) \leq f_G(m', n') = t \cdot \|(m', n')\|. \tag{10}$$

Hence $\|(n, -m)\|_* \leq \frac{4}{3}$, and therefore, $(n, -m)$ belongs to $\frac{4}{3} \cdot K^*$. This shows Theorem 2(i).

Again, any better factor in Theorem 1(i) would imply a better factor in Theorem 2(i). This gives the implication Theorem 2(ii) \Rightarrow Theorem 1(ii). ■

In Section 2 we will prove Theorem 4 and develop some further results on integer norms in relation to graphs on the torus, and in Sections 3 we give a proof of Theorem 2.

2. INTEGER NORMS AND GRAPHS ON THE TORUS

In this section we give a proof of Theorem 4 above. To this end, we derive some further results. The following theorem follows directly from the “cutting plane theorem” of Chvátal [5]. It is a slight extension of a result of Hoffman [10] for polytopes (extended by Edmonds and Giles [8] to polyhedra, forming the basis for the theory of *total dual integrality*—cf. [18, Chap. 23]).

A *polytope* is the convex hull of a finite set of vectors. A polytope P is called *integer* if each vertex of P is an integer vector.

THEOREM 5. *Let C be a nonempty compact convex set in \mathbb{R}^n . Then C is an integer polytope, if and only if $\max\{c^T x \mid x \in C\}$ is an integer for each integer vector $c \in \mathbb{R}^n$.*

This implies:

THEOREM 6. *For any integer norm $\|\cdot\|$ in \mathbb{R}^n there exist integer vectors y_1, \dots, y_t in \mathbb{R}^n such that for each $x \in \mathbb{R}^n$:*

$$\|x\| = \max\{y_1^T x, \dots, y_t^T x\}. \quad (11)$$

Proof. Let $K := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Hence for each $x \in \mathbb{R}^n$:

$$\|x\| = \max\{x^T y \mid y \in K^*\}. \quad (12)$$

As $\|\cdot\|$ is an integer norm, this maximum is an integer for each integer vector x . Hence, by Theorem 5, K^* is an integer polytope. So we can take for y_1, \dots, y_t the vertices of K^* . ■

Remark 3. One similarly shows the following related result. Any function $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}_+$ satisfies

$$\begin{aligned} \text{(i)} \quad & \varphi(x + x') \leq \varphi(x) + \varphi(x') \quad \text{for all } x, x' \in \mathbb{Z}^n, \\ \text{(ii)} \quad & \varphi(k \cdot x) = |k| \cdot \varphi(x) \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{Z}^n, \end{aligned} \quad (13)$$

if and only if there exist integer vectors y_1, \dots, y_t such that

$$\varphi(x) = \max\{|y_1^T x|, \dots, |y_t^T x|\} \quad (14)$$

for each $x \in \mathbb{Z}^n$.

Equivalently, in terms of groups: Let G be an abelian group. Then any function $\varphi: G \rightarrow \mathbb{Z}_+$ satisfies

$$\begin{aligned} \text{(i)} \quad & \varphi(x + x') \leq \varphi(x) + \varphi(x') \quad \text{for all } x, x' \in G, \\ \text{(ii)} \quad & \varphi(k \cdot x) = |k| \cdot \varphi(x) \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in G, \end{aligned} \quad (15)$$

if and only if there exist homomorphisms $\varphi_1, \dots, \varphi_t: G \rightarrow \mathbb{Z}$ such that

$$\varphi(x) = \max\{|\varphi_1(x)|, \dots, |\varphi_t(x)|\} \quad (16)$$

for each $x \in G$.

For integer norms in \mathbb{R}^2 we derive from Theorem 6 a further characterization.

THEOREM 7. *A norm $\|\cdot\|$ in \mathbb{R}^2 is integer, if and only if there exist integer vectors z_1, \dots, z_k in \mathbb{R}^2 such that*

$$\|x\| = \frac{1}{2} \sum_{i=1}^k |z_i^T x| \quad (17)$$

for each $x \in \mathbb{R}^2$ and such that both components of the vector $z_1 + \dots + z_k$ are even.

Proof. Sufficiency of the condition follows from the fact that, for any integer vector x ,

$$\frac{1}{2} \sum_{i=1}^k z_i^T x = \left(\frac{1}{2}(z_1 + \dots + z_k)\right)^T x \tag{18}$$

is an integer that differs by an integer value, viz.

$$\sum_{i=1}^k \frac{1}{2} (|z_i^T x| - z_i^T x), \tag{19}$$

from $\|x\|$ (by (17)).

To see necessity, let $K := \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$. By Theorem 6, K^* is a polygon with integer vertices, y_1, \dots, y_{2k} , say, in cyclic order. So $y_{i+k} = -y_i$ for $i = 1, \dots, k$. Define

$$z_i := y_{i+1} - y_i \tag{20}$$

for $i = 1, \dots, k$. So $z_1 + \dots + z_k = y_{k+1} - y_1 = 2y_{k+1}$ is an even vector. We show that (17) holds for each $x \in \mathbb{R}^2$.

Since $\|x\| = \max\{y^T x \mid y \in K^*\}$, we know that

$$\|x\| = \max\{y_1^T x, \dots, y_{2k}^T x\}. \tag{21}$$

Let the maximum be attained by $y_j^T x$. Without loss of generality, $1 \leq j \leq k$. It follows that $z_1^T x, \dots, z_{j-1}^T x \geq 0$ and $z_j^T x, \dots, z_k^T x \leq 0$. Hence

$$\sum_{i=1}^k |z_i^T x| = \sum_{i=1}^{j-1} z_i^T x - \sum_{i=j}^k z_i^T x. \tag{22}$$

Now $y_1 = -y_{k+1} = -\frac{1}{2}(z_1 + \dots + z_k)$, implying that $y_j = y_1 + z_1 + \dots + z_{j-1} = \frac{1}{2}(z_1 + \dots + z_{j-1} - z_j - \dots - z_k)$. Hence the right-hand side of (22) is equal to $2y_j^T x = 2\|x\|$. ■

We are now able to prove:

THEOREM 4. For each integer norm $\|\cdot\|$ in \mathbb{R}^2 there exists a graph G embedded on the torus such that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n) .

Proof. Let $\|\cdot\|$ be an integer norm in \mathbb{R}^2 . By Theorem 7, there exist integer vectors z_1, \dots, z_k in \mathbb{R}^2 such that

$$\|x\| = \frac{1}{2} \sum_{i=1}^k |z_i^T x| \tag{23}$$

holds for each $x \in \mathbb{R}^2$ and such that $z_1 + \dots + z_k$ is an even vector.

We may assume that, for each $i = 1, \dots, k$ the two components of z_i are relatively prime (as z_1, \dots, z_k need not all be different). Write $z_i = (z'_i, z''_i)^T$ for $i = 1, \dots, k$.

Again, let $S = S^1 \times S^1$ be the torus. Let $\Pi: \mathbb{R}^2 \rightarrow S$ be the usual projection of \mathbb{R}^2 to the torus (i.e., $\Pi(x) := (e^{2\pi i x'}, e^{2\pi i x''})$ for each $x = (x', x'')^T$ in \mathbb{R}^2). Call a simple closed curve D on S *geodesic* if each component of $\Pi^{-1}[D]$ is a straight line in \mathbb{R}^2 .

For each $i = 1, \dots, k$, let D_i be a geodesic simple closed curve on S freely homotopic to $C_{z'_i, -z''_i}$, in such a way that each two of the D_i are different and no point of S is in more than two of the D_i . So $\text{mincr}(C_{m,n}, D_i) = |mz'_i + nz''_i|$ for all m, n .

Let H be the four-regular graph on the torus formed by the union of D_1, \dots, D_k . Then one easily checks that for each $(m, n) \in \mathbb{Z}^2$:

each closed curve C freely homotopic to $C_{m,n}$, not traversing vertices of H , has at least

$$\sum_{i=1}^k \text{mincr}(C_{m,n}, D_i) = \sum_{i=1}^k |mz'_i + nz''_i| = 2\|(m, n)\|$$

crossings with H ; moreover, at least one such curve has exactly this number of crossings with H . (24)

(Indeed, C has at least $\text{mincr}(C_{m,n}, D_i)$ crossings with part D_i of H . This gives the lower bound. Equality can be attained by $C_{m,n}$ itself or a slight shift of it.)

Since $z_1 + \dots + z_k$ is even, we know that each closed curve on S , not traversing vertices of H , has an even number of crossings with H . So we can color each face of H black or white in such a way that adjacent faces have different colors.

Hence we can construct a "radial" graph G as follows: In each black face F , put a vertex and connect it by (pairwise disjoint) lines through F to each of the vertices of H incident with F . Doing this for each black face of H , we obtain a graph G , called a *radial* graph G .

Now each closed curve on S , intersecting H r times and not intersecting vertices of H , can be shifted slightly so that it intersects G $\frac{1}{2}r$ times (in

vertices of G). So from (24) we have that $f_G(m, n) = \|(m, n)\|$ for each integer vector (m, n) . ■

Remark 4. The graph G in Theorem 4 need not be unique, but (as was shown in [20]) the minimal such graphs are unique, in the following sense:

Let G be a graph embedded on the torus S . A *minor* of G is any graph obtained from G by a series of deletions and contractions of edges (contracting loops only if they enclose a face). Any minor of G has a natural embedding on S derived from the embedding of G . It is a *proper* minor if at least one edge is deleted or contracted. Call a graph G embedded on the torus S a *kernel* if for each proper minor G' of G one has $f_{G'} \neq f_G$ (i.e., $f_{G'}(m, n) < f_G(m, n)$ for at least one integer vector (m, n)).

So for each integer norm $\|\cdot\|$ in \mathbb{R}^2 there exists at least one kernel G on S with $f_G(m, n) = \|(m, n)\|$ for all integer vectors (m, n) . Now by the results in [20], for any fixed integer norm $\|\cdot\|$, each two such kernels can be obtained from each other by a series of the following operations:

- (i) shifting the graph over the torus;
- (ii) taking the surface dual of the graph; (25)
- (iii) ΔY -exchange.

Here ΔY -exchange means replacing a triangular face F by a vertex in the face connected to the three vertices incident with F , or conversely. (This operation was introduced by Steinitz [22], who called it the θ -process.)

3. PROOF OF THEOREM 2

Although Theorem 2 is nothing but a simple exercise in plane geometry, for completeness we give a proof here. As a preparation, we first give another simple fact.

THEOREM 8. *For any nonsingular 2×2 matrix A there exist nonzero integer vectors x and y in \mathbb{R}^2 such that*

$$\|Ax\|_\infty \cdot \|y^T A^{-1}\|_1 \leq \frac{1}{2}(\sqrt{2} + 1). \tag{26}$$

Proof. We may assume that $\det A = 1$. Let A and A^* be the pair of dual lattices

$$A := \{Ax \mid x \in \mathbb{Z}^2\}, \quad A^* := \{y^T A^{-1} \mid y \in \mathbb{Z}^2\}. \tag{27}$$

We may assume that A has a basis $b = (b_1, b_2)^T$, $c = (c_1, c_2)^T$ satisfying

$$b_1 \geq b_2 \geq 0 \quad \text{and} \quad c_2 \geq -c_1 \geq 0. \tag{28}$$

Indeed, let b be a nonzero vector in A minimizing $\|b\|_\infty$. Without loss of generality, $\|b\|_\infty = b_1 \geq b_2 \geq 0$. Let c be a nonzero vector in A minimizing $|c_2|$ over all nonzero vectors $c \in A$ with $|c_1| \leq \|b\|_\infty$. We may assume that $c_2 \geq 0$, and that the triangle Δ with vertices 0 , b , and c has minimal area. If b and c do not form a basis, Δ would contain another vector c' with the required properties, contradicting the minimality of Δ . Moreover, $\|c\|_\infty = c_2 \geq \|b\|_\infty > |c_1|$. If $c_1 > 0$, we can replace c by $c - b$. Thus we obtain b and c satisfying (28).

The arithmetic-geometric inequality $\alpha\beta \leq (\frac{1}{2}\alpha + \frac{1}{2}\beta)^2$, applied to $\alpha = (\sqrt{2} - 1)b_1c_2$, $\beta = -(\sqrt{2} + 1)b_2c_1$, and the fact that $b_1c_2 - b_2c_1 = \det A = 1$ give

$$-b_1b_2c_1c_2 \leq (\frac{1}{2}(\sqrt{2} - 1)b_1c_2 - \frac{1}{2}(\sqrt{2} + 1)b_2c_1)^2 = (\frac{1}{2}(\sqrt{2} + 1) - b_1c_2)^2. \tag{29}$$

Now $\frac{1}{2}(\sqrt{2} + 1) - b_1c_2 \geq 1 - b_1c_2 = -b_2c_1 \geq 0$. Hence at least one of b_2c_2 and $-b_1c_1$ is at most $\frac{1}{2}(\sqrt{2} + 1) - b_1c_2$. That is, at least one of $(b_1 + b_2)c_2$ and $b_1(-c_1 + c_2)$ is at most $\frac{1}{2}(\sqrt{2} + 1)$. Since b and c belong to A and since $(b_2, -b_1)$ and $(c_2, -c_1)$ belong to A^* , we have the required vectors. ■

In fact, bound $\frac{1}{2}(\sqrt{2} + 1)$ in Theorem 8 is best possible, as is shown by the matrix

$$A = \begin{pmatrix} 1 & 1 - \sqrt{2} \\ \sqrt{2} - 1 & 1 \end{pmatrix}. \tag{30}$$

THEOREM 2. (i) *For any symmetric convex body K in \mathbb{R}^2 , there exists a nonzero integer vector in K or there exists a nonzero integer vector in $\frac{4}{3} \cdot K^*$.*

(ii) *The factor $\frac{4}{3}$ is best possible.*

Proof. (i) We may assume that K is a polygon. We show that if K contains no nonzero integer vectors in its interior, then $\frac{4}{3}K^*$ contains a nonzero integer vector.

We may assume that each edge of K contains an integer vector in its relative interior (otherwise, we can shift the edge until it contains an integer vector in its relative interior or until the edge “disappears”).

If K has four edges, the result directly follows from Theorem 8 (applied to the matrix A with rows the coefficients of the inequalities determining the edges of K (taking for each two parallel edges one of the two)), since $\frac{1}{2}(\sqrt{2} + 1) < \frac{4}{3}$.

If K has at least six edges, let v_1, \dots, v_{2k} be the vertices of K (in cyclic order), and let z_i be an integer vector in the relative interior of the edge connecting v_{i-1} and v_i ($i = 1, \dots, 2k$, taking indices mod $2k$).

By Minkowski's theorem [13], the volume of K is at most 4. Hence, there exists an $i = 1, \dots, 2k$, so that the volume V of the quadrangle $(0, z_i, v_i, z_{i+1})$ is at most $4/2k$. As the triangle $(0, z_i, z_{i+1})$ contains no further integer vectors, z_i and z_{i+1} form a basis for the lattice \mathbb{Z}^2 . So the vector c satisfying $c^T z_i = c^T z_{i+1} = 1$ is an integer vector. Let V_1 and V_2 be the volumes of the triangles $(0, z_i, z_{i+1})$ and (z_i, z_{i+1}, v_i) , respectively. So $V_1 = \frac{1}{2}$ and $V_2 = V - V_1$. Moreover, $V_2/V_1 = (c^T v_i - c^T z_i)/c^T z_i$. This implies $c^T v_i = 2V$. Hence

$$\max\{c^T x \mid x \in K\} = c^T v_i = 2V \leq 2 \frac{4}{2k} \leq \frac{4}{3}. \tag{31}$$

(ii) Let K be the convex hull of the vectors $\pm(\frac{2}{3}, \frac{4}{3}), \pm(\frac{4}{3}, \frac{2}{3}), \pm(-\frac{2}{3}, \frac{2}{3})$. Then K^* is the convex hull of the vectors $\pm(-\frac{1}{2}, 1), \pm(1, -\frac{1}{2}), \pm(\frac{1}{2}, \frac{1}{2})$. Since no slight shrinking of K and of $\frac{4}{3}$, K^* contains any nonzero integer vector, we obtain that $\frac{4}{3}$ is best possible. ■

Remark 5. In fact, in this proof k cannot exceed 3, as no two of the vectors z_i and $z_{i'}$ for $i, i' = 1, \dots, k$, are equal mod 2 (otherwise $\frac{1}{2}(z_i + z_{i'})$ would be an integer vector in the interior of K). (This is a special case of a result of Doignon [7] (cf. [2, 17]).)

ACKNOWLEDGMENTS

I am grateful to three anonymous referees and to Bill Cook and Paul Seymour for helpful suggestions.

REFERENCES

1. D. ARCHDEACON, Densely embedded graphs, *J. Combin. Theory Ser. B* **54** (1992), 13–36.
2. D. E. BELL, A theorem concerning the integer lattice, *Stud. Appl. Math.* **56** (1977), 187–188.
3. R. BRUNET, B. MOHAR, AND R. B. RICHTER, Disjoint homotopic essential cycles, preprint, 1992.
4. J. W. S. CASSELS, "An Introduction to the Geometry of Numbers," Springer-Verlag, Berlin, 1959.
5. V. CHVÁTAL, Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Math.* **4** (1973), 305–337.
6. G. DING, A. SCHRIJVER, AND P. D. SEYMOUR, Disjoint cycles in directed graphs on the torus and the Klein bottle, *J. Combin. Theory Ser. B*, to appear.
7. J.-P. DOIGNON, Convexity in crystallographical lattices, *J. Geom.* **3** (1973), 71–85.
8. J. EDMONDS AND R. GILES, A min–max relation for submodular functions on graphs, *Ann. Discrete Math.* **1** (1977), 185–204.
9. J. R. FIEDLER, J. P. HUNEKE, R. B. RICHTER, AND N. ROBERTSON, Computing the orientable genus of projective graphs, preprint, 1989.

10. A. J. HOFFMAN, A generalization of max flow-min cut, *Math. Programming* **6** (1974), 352–359.
11. C. G. LEKKERKERKER, “Geometry of Numbers,” North-Holland, Amsterdam, 1969.
12. K. MAHLER, On lattice points in polar reciprocal convex domains, *Indag. Math.* **10** (1948), 176–179.
13. H. MINKOWSKI, Extrait d’une lettre adressée à M. Hermite, *Bull. Sci. Math. (2)* **17** (1893), 24–29, reprinted in “Gesammelte Abhandlungen von Hermann Minkowski, Band II” (D. Hilbert, Ed.), pp. 266–270, Teubner, Leipzig, 1911.
14. N. ROBERTSON AND P. D. SEYMOUR, Graph minors. VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* **45** (1988), 212–254.
15. N. ROBERTSON AND R. THOMAS, On the orientable genus of graphs embedded in the Klein bottle, *J. Graph Theory* **15** (1991), 406–419.
16. N. ROBERTSON AND R. VITRAY, Representativity of surface embeddings, in “Paths, Flows, and VLSI-Layout” (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, Eds.), pp. 293–328, Springer-Verlag, Berlin, 1990.
17. H. E. SCARF, An observation on the structure of production sets with indivisibilities, *Proceedings, National Academy of Sciences of the United States of America* **74** (1977), 3637–3641.
18. A. SCHRIJVER, “Theory of Linear and Integer Programming,” Wiley, Chichester, 1986.
19. A. SCHRIJVER, Disjoint circuits of prescribed homotopies in a graph on a compact surface, *J. Combin. Theory Ser. B* **51** (1991), 127–159.
20. A. SCHRIJVER, On the uniqueness of kernels, *J. Combin. Theory Ser. B* **55** (1992), 146–160.
21. P. D. SEYMOUR, Directed circuits on the torus, *Combinatorica* **11** (1991), 261–273.
22. E. STEINITZ, Polyeder und Raumeinteilungen, in “Encyclopädie der mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen, Dritter Band (Geometrie), Erster Teil, Zweite Hälfte” (W. F. Meyer and H. Mohrmann, Eds.), Part IIAB12, pp. 1–139, Teubner, Leipzig, 1922.
23. J. STILLWELL, “Classical Topology and Combinatorial Group Theory,” Springer-Verlag, New York, 1980.
24. C. THOMASSEN, Embeddings of graphs with no short noncontractible cycles, *J. Combin. Theory Ser. B* **13** (1989), 141–148.